Averages associated to the energy momentum tensor of new geometries describing dark matter

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Content

Introduction I

In this short talk we will give a synthetic description of a line of research intended to understand the problems involved in the study of systems with dark matter.
We will have in mind an intermediate size system as a cluster of galaxies.

The two principal tools used in the description of the phenomena of DM are:

- The geodesic equation for massive particles

$$v^a \nabla_a v^b = v^a \partial_a v^b + \Gamma_{a\ c}^{\ b} v^a v^c = 0; \tag{1}$$

  which is needed to deal with issue of rotation curves and the dynamics of galaxies in clusters.

- The geodesic deviation equation for massless particles

$$\ell^a \nabla_a \left(\ell^b \nabla_b \chi^d \right) = -R_{abc}{}^{d} \ell^a \chi^b \ell^c; \tag{2}$$

  from which one can get the formalism of gravitational lensing.

Introduction II

Recently background work:

We have at hand more general expression for the bending angle and optical scalars and new geometries describing some observational aspects of the dark matter (DM) phenomena [GM11, GM12].

Non-conventional T_{ab}:

These works stress the fact that the field equations make use of the complete energy momentum tensor, namely

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}. \quad (3)$$

Therefore, non-conventional energy-momentum tensor, T_{ab}, might have a non-trivial application to the description of dark matter phenomena.

In particular, this point out to the importance of <u>space-like components</u> of T_{ab}.

It might be that a proper understanding of the averaging problem can be interpreted as providing new components of the energy momentum tensor.

Averages in general relativity I

A context in which extra contributions to the matter-energy terms arises is in the discussion of averages in systems with many scales.

Examples:

- The high frequency limit [Isa68b, Isa68a, Bur89].
- Averages have been used recently in the problem of back-reaction associated to small scales inhomogeneities in the universe as a possible contribution to dark energy [Ell09, Ell11, GW11, ea11].

- One arrives to an average geometry with satisfies a modified Einstein's equation with an additional matter source arising after averaging;

$$R_{ab}\left(\bar{g}\right) - \frac{1}{2} R\left(\bar{g}\right) \bar{g}_{ab} = \kappa \bar{T}_{ab} + \kappa T^{(\text{eff})}_{ab}. \tag{4}$$

Averages in general relativity II

- In general, if one has at hand a definition of averages in terms of an operator " $\bar{\cdot}$ ", after averaging the field equations one obtains

$$\bar{R}_{ab} - \frac{1}{2}\bar{Rg}_{ab} = \kappa \bar{T}_{ab}, \quad (5)$$

and writing the averages in terms of the curvature quantities, $R_{ab}(\bar{g})$ and $R(\bar{g})$, associated to $\bar{g}_{ab}$,

$$\bar{R}_{ab} = R_{ab}(\bar{g}) + Z_{ab}, \quad \text{and} \quad \bar{Rg}_{ab} = R(\bar{g})\bar{g}_{ab} + Y_{ab}, \quad (6)$$

results

$$R_{ab}(\bar{g}) - \frac{1}{2}R(\bar{g})\,\bar{g}_{ab} = \kappa \bar{T}_{ab} + \kappa T^{(\mathrm{eff})}_{ab}, \quad (7)$$

where

$$\kappa T^{(\mathrm{eff})}_{ab} = -\left(Z_{ab} - \frac{1}{2}Y_{ab}\right). \quad (8)$$

Issues in taking averages I

There exists limitations when one deals with averages notions.

Practical implementation:

- In general average is a linear operation that involves the integration of a given quantity $\mathcal{Q}$ on an specified domain $\mathcal{V}$;

$$\bar{\mathcal{Q}} \propto \int_{\mathcal{V}} \mathcal{Q}\, d\mathcal{V}. \tag{9}$$

- This notion is well defined for quantities on vectorial spaces but in the context of curved manifolds integration is only well defined for scalar quantities.
- Tensor fields as g_{ab}, ${C_{abc}}^{d}$, R_{ab} or T_{ab} can not be put in a naive definition of average since tensors can not be added at two different points.

Issues in taking averages II

Physical considerations: Cosmology

- Averages are used in a non very well understood sense in cosmology.
- The universe is a coarse-grained description of an irregular matter distribution of structures presents in many scales.

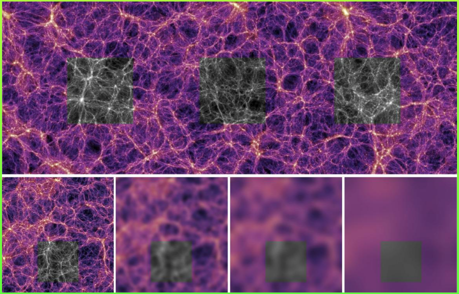

Figure: C. Clarkson *et al.* Rep. Prog. Phys. **74** (2011) 112901

Issues in taking averages III

Physical considerations: Cosmology

- Typically, one assumes that the geometry is well described by the Robertson-Walker metrics

$$ds^2 = d\tau^2 - a^2(\tau)\left[\frac{dr^2}{1+kr^2} + r^2 d\Omega^2_{S^2}\right], \qquad k = 1, 0, -1; \tag{10}$$

when increasingly large volume are considered.

- In a first approach one might consider the distribution of matter to be determined by the distribution of galaxies. If matter were only concentrated with visible matter; then at an intermediate point between galaxies, one would have a vanishing T_{ab} but an expected non-zero curvature.
For most points in space they would satisfies

$$R_{ab} = 0 \qquad \text{but} \qquad C_{abc}{}^{d} \neq 0. \tag{11}$$

This is the case from the point of view of observations where light rays travel through empty space, $R_{ab} = 0$, although shear effects are present $C_{abc}{}^{d} \neq 0$.

Issues in taking averages IV

Physical considerations: Cosmology

- This is totally opposite to the Robertson-Walker case, in which one has

$$R_{ab} \neq 0 \qquad \text{but} \qquad {C_{abc}}^{d} = 0. \tag{12}$$

- Some authors have been emphasizing the needed for a more rigorous justification when one goes to the continuous limit. [GW11, Kor13]

This implies the needed for a deeper understanding of averages.

What one should be average?

The study of DM with new geometries

- Massive particles test the connection $\Gamma_{a\ c}^{\ b}$ through the geodesic equation

$$v^a \nabla_a v^b = v^a \partial_a v^b + \Gamma_{a\ c}^{\ b} v^a v^b = 0. \tag{13}$$

- Massless particles test the curvature through the geodesic deviation equation

$$\ell^a \nabla_a \left(\ell^b \nabla_b \chi^d \right) = -R_{abc}{}^d \ell^a \chi^b \ell^c. \tag{14}$$

The relevant curvature quantities in the calculation of the optical scalars are:

$$\Phi_{00} = -\frac{1}{2} R_{ab} \ell^a \ell^b, \quad \Psi_0 = C_{abcd} \ell^a m^b \ell^c m^d. \tag{15}$$

One is the component of the Ricci tensor and the other a component of the Weyl tensor.

A system with two scales II

We want to consider a system composed of small subsystems that contribute to a big completed system where each subsystem is considered as an spherically symmetric gravitating object.

- Each subsystem is considered to have very small velocity with respect to the others so that all of them can be considered geometric linear contribution over a common flat background.
- The distribution of the small subsystems can be described in terms of the stationary and spherically symmetric distribution function $\Pi(x^i)$, with $i = 1, 2, 3$ denoting the space-like coordinates of the flat background.
- We are assuming that the nature of the observations is such that one can consider each subsystem and the compound system as stationary; so that we can assume the existence of a global Killing vector K_t.

A system with two scales III

Decomposition of the geometry Since the distribution characterizing the big system is assumed spherically symmetric, the line element can be cast in the form:

$$ds^2 = e^{2\Phi(r)}dt^2 - \frac{dr^2}{1 - \frac{2\,M(r)}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (16)$$

One must keep in mind that this corresponds to an averaged geometry. Where the metric can be thought in terms of the reference metric η_{ab}:

$$g_{ab} = \eta_{ab} + h_{ab}. \qquad (17)$$

Due to the fact that one is considering contributions of many subsystems α;

$$h_{ab} = \sum_{[\alpha]} h_{[\alpha]ab}. \qquad (18)$$

Testing the system with massless particles.
For thin lenses the bending angle is given by [GM11]

$$\alpha(J) = J\left(\hat{\Phi}_{00}(J) + \hat{\Psi}_0(J)\right); \tag{19}$$

$$\hat{\Phi}_{00} = \int \Phi_{00} d\lambda, \qquad \hat{\Psi}_0 = \int \Psi_0 d\lambda. \tag{20}$$

This expression are valid for each subsystem.

- The curvature scalar are also function of the components of the microscopic energy-momentum tensor; p_r, p_t, ρ and m.
- For the big system we will obtain similar expressions but containing effective curvature quantities depending on the distribution;

$$\alpha^{(\text{eff})}(J) = J\left(\hat{\Phi}_{00}^{(\text{eff})}(J) + \hat{\Psi}_0^{(\text{eff})}(J)\right); \tag{21}$$

A system with two scales V

This will involve macroscopic components of the energy-momentum tensor, P_r, P_t, ϱ, and $M(r)$.

- This constitutes a generalization of the standard treatment of thin lenses in which one considers the small subsystems as Schwarzschild contributions

$$\hat{\Phi}_{00} = 0, \qquad \hat{\Psi}_0 = \frac{4m}{J^2}, \qquad \alpha(J) = \frac{4m}{J}; \tag{22}$$

and the big system a linear superposition of them [SEF92].

A system with two scales VI

Other possible microscopic model:

- If the small subsystems where all point like objects, then one would proceed with the standard approach in which the bending angle can be calculated in terms of the projected masses on the plane of the lens.
- However, if the small systems have contributions not only from the mass energy density component ρ, but also from spacelike components as P_r, then one can not use the standard expressions.

Testing a two scale system with massive particles
We assume small velocities

$$v^a \approx \left(1 + \frac{v^2}{2}\right) K_t^a + v_\perp^a; \tag{23}$$

$$0 = \eta_{ab} K_t^a v_\perp^a; \tag{24}$$

$$-v^2 = \eta_{ab} v_\perp^a v_\perp^b. \tag{25}$$

The geodesic equation at first order in the velocities results

$$K_t^a \partial_a v_\perp^b + \Gamma_{a\ c}^{\ b} K_t^a K_t^c + 2\Gamma_{a\ c}^{\ b} K_t^a v_\perp^c = 0; \tag{26}$$

A system with two scales VIII

In a weak field regime one has

$$\Gamma^{r}_{\theta\,\theta} = 2M(r), \tag{27}$$

$$\Gamma^{r}_{\phi\,\phi} = 2M(r)\sin^2\theta, \tag{28}$$

$$\Gamma^{t}_{t\,r} = \frac{d\Phi(r)}{dr}, \tag{29}$$

$$\Gamma^{r}_{r\,r} = \frac{d}{dr}\left(\frac{M(r)}{r}\right). \tag{30}$$

$$\boxed{\Gamma^{r}_{t\,t} = \frac{d\Phi(r)}{dr}}, \tag{31}$$

A system with two scales IX

The only dynamically interesting equation is

$$\frac{dv_\perp^r}{dt} = -\frac{d\Phi}{dr}. \tag{32}$$

- $\Phi(r)$ contains the information of the small subsystems in a statistical sense. Since the effective equation of motion, at the lowest order, is the Newtonian equation, and the big scale distribution is assumed spherically symmetric, one can use Newton theorem on spherical systems and evaluate $\Phi(r)$ from the distribution inside the sphere o radius r.

Final comments I

- We have reviewed the problems associated with naive notions of averages, as are used in cosmology. We have presented elsewhere a definition of tensors along with the intrinsic notion of derivative of averages; providing a technique for concrete studies.
- We have just seen that when studying the dynamics of massive and massless particles, while the first reduces to the simple application of Newtonian techniques, the later is more complicated if the spacelike components of the energy momentum tensor can not be neglected.
- We plan to complete and extend these studies to the cosmological framework, and to find an insight in the way in which the average process appears and is used in general relativity.
- In the future we also will consider other geometries for the description of the small subsystems by including the spheroidal geometries mentioned in the previous talk and also by including spatial components of the energy-momentum tensor for the matter distribution.

Thanks!

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