# *Validity Conditions for Linear Cosmological Perturbations*

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**GrACo2** - *Buenos Aires 2014*

# Planck and WMAP

- Standard Model: $\Lambda$CDM in a FLRW universe, structure formation, nucleosynthesis, etc.
- Primordial spectrum[1]
  - Planck $\oplus$ WMAP gives $n_s = 0.9603 \pm 0.0073$, ruling out exact scale invariance at $5\sigma$
  - No significant running.... $\frac{dn_s}{d \ln k} = 0.0134 \pm 0.0090$
- Cosmological Parameters[2]
  - $H_0 = 67.3 \pm 1.2 Km/s^{-1} Mpc^{-1}$
  - $\Omega_b.h^2 = 0.02205 \pm 0.00028$
  - $\Omega_m = 0.315 \pm 0.017$
- These values agrees with BAO but appears a small tension with Type Ia supernovae
- $BAO\ and\ CMB \neq SNe\ Ia$ wider light beams[3]

$$5\ arcmin \times 10^{-7} arcsec$$

[1] *astro-ph.CO* 1303.5082 - Planck 2013 results. XXII. Constraints on inflation
[2] *astro-ph.CO* 1303.5076 - Planck 2013 results. XVI. Cosmological parameters
[3] *astro-ph.CO* 1304.7791 - Fleury P., Dupuy H., Uzan J-P.,

# Validity of Linear Perturbations

## Why bother and all that jazz...

- Primordial Universe
  - linearity through a bouncing phase [4] [5]
- Late Universe
  - In small scale is definitely (very!) inhomogeneous
  - Is dark energy related to back-reaction effects (cosmic coincidence)?

defining the average $\bar{g}_{\mu\nu} \equiv \langle g_{\mu\nu} \rangle$, we have $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

$\bar{g}^{\mu\nu} \neq \langle g^{\mu\nu} \rangle$ but still $\bar{\Gamma}^{\alpha}_{\mu\nu}$ and $\bar{R}^{\alpha}{}_{\beta\mu\nu}$

$$\langle R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R \rangle = \langle g^{\mu\alpha} R_{\alpha\nu} \rangle - \frac{1}{2}\delta^{\mu}{}_{\nu} \langle g^{\alpha\beta} R_{\alpha\beta} \rangle = \frac{8\pi G_{\mathrm{N}}}{c^4} \langle T^{\mu}{}_{\nu} \rangle$$

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = \frac{8\pi G_{\mathrm{N}}}{c^4} \langle T^{\mu}{}_{\nu} \rangle \quad ,$$

[4] S.D.P. Vitenti, F.T. Falciano, N. Pinto-Neto - *Phys. Rev.* **D 87**, 103503 (2013)

[5] F.T. Falciano, N. Pinto-Neto, S.D.P. Vitenti - *Phys. Rev.* **D 87**, 103514 (2013)

## Cosmological Perturbations

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Which one is the best approach?

- Bardeen's
  - expansion of the components of the metric

$$ds^2 = -(1-2\phi)dt^2 - 2\mathcal{B}_{\|i}dt\,dx^i + \left((1+2\psi)\bar{\gamma}_{ij} - 2\mathcal{E}_{\|ij}\right)dx^i dx^j$$

$$g_{00} = \bar{g}_{00} + 2\phi = -1 + 2\phi$$

  - gauge issues.... $\delta\rho^{(\mathrm{gi})} = \delta\rho + \dot{\bar{\rho}}\left(\mathcal{B} - \dot{\mathcal{E}}\right)$

- Covariant Approach
  - based on Stewart-Walker lemma
  - gauge invariant variables (vanish in the background)

$$a^\mu\ , \quad \sigma_{\mu\nu}\ , \quad E^{\mu\nu}\ , \quad H^{\mu\nu}$$

$$X_\mu = \kappa D_\mu \rho\ , \quad Y_\mu = \kappa D_\mu p\ , \quad Z_\mu \equiv D_\mu \Theta$$

  - foliation dependent....

- both are equivalent.... mixed $\otimes$ pure tensors[6]

[6] Falciano, F.T., Vitenti, S.D.P. and Pinto-Neto, N. - arXiv:1311.6730 [astro-ph.CO]

## Cosmological Perturbations



### Linearizing the system...

- assumption: $\delta g_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ is small
- physical spacetime $g_{\mu\nu}$ and a background FLRW $\bar{g}_{\mu\nu}$

$$\delta g_{\mu\nu} = 2\phi \bar{v}_\mu \bar{v}_\nu + 2B_{(\mu}\bar{v}_{\nu)} + 2C_{\mu\nu},$$

where

$$\phi \equiv \frac{1}{2}\delta g_{\mu\nu}\bar{v}^\mu \bar{v}^\nu \quad , \quad B_\mu \equiv -\bar{\gamma}\left[\delta g_{\mu\nu}\bar{v}^\nu\right] = \mathcal{B}_{\|\mu} + \mathtt{B}_\mu$$

$$C_{\mu\nu} \equiv \frac{1}{2}\bar{\gamma}\left[\delta g_{\mu\nu}\right] = \psi\bar{\gamma}_{\mu\nu} - \mathcal{E}_{\|\mu\nu} + \mathtt{F}_{(\mu\|\nu)} + W_{\mu\nu}$$

- physical foliation.... $v^\mu = (1-\phi)\bar{v}^\mu + \overline{D}_\mu \mathcal{V}$
- perturbed quantities.....

$$\delta\mathring{a}_\mu = \dot{\mathcal{V}}_{\|\mu}$$

$$\delta\mathring{\sigma}_{\mu\nu} = (\delta\sigma^{\mathsf{s}} + \mathcal{V})_{\|\langle\mu\nu\rangle} \qquad \delta\sigma^{\mathsf{s}} \equiv \left(\mathcal{B} - \dot{\mathcal{E}} + \frac{2}{3}\overline{\Theta}\mathcal{E}\right)$$

$$\delta\mathring{\Theta} = \overline{D}^2\delta\sigma^{\mathsf{s}} + \overline{\Theta}\phi + 3\dot{\psi} + \overline{D}^2\mathcal{V}$$

- Five perturbed quantities..... $\{\phi\,,\ \psi\,,\ \mathcal{E}\,,\ \mathcal{B}\,,\ \mathcal{V}\} \oplus \delta T^{\mu\nu}$

## Cosmological Perturbations

still linearizing...

- Einstein Equations

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = \kappa \left( \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \right) \quad ,$$

- Linearity requires that $|\delta G_{\mu\nu}| \ll |\bar{G}_{\mu\nu}|$ and $|\delta T_{\mu\nu}| \ll |\bar{T}_{\mu\nu}|$

$$\bar{G}^0{}_0 + \delta G^0{}_0 = -\left( \frac{\overline{\Theta}^2}{3} + \frac{3\overline{K}}{a^2} \right) - \frac{2\overline{\Theta}\delta\Theta}{3} - \frac{\delta\mathcal{R}}{2}$$

with $\quad \delta\mathcal{R} = -\frac{4}{a^2}\left( \overline{D}^2 + 3\overline{K} \right)\psi$

hence

$$\left| \frac{\delta\Theta}{\overline{\Theta}} \right| \ll 1 \quad \text{and} \quad \left| \frac{\left( \overline{D}^2 + 3\overline{K} \right)\psi}{\overline{K}} \right| \ll 1$$

## Cosmological Perturbations

### First order conditions[7]

- perturbations are small.... $|\delta g_{\mu\nu}| \ll 1$

$$|\phi| \ll 1 \ , \ |\psi| \ll 1 \ , \ |\overline{D}_i B| \ll 1 \ , \ |\overline{D}_{ij}\mathcal{E}| \ll 1 \ , \ |\overline{D}_i \mathcal{V}| \ll 1$$

- in addition

$$i) \left|\frac{\delta\Theta}{\overline{\Theta}}\right| \ll 1 \quad , \quad ii) \left|\frac{\delta\rho}{\bar{\rho}}\right| \ll 1 \quad , \quad iii) \left|\frac{\delta p}{\bar{p}}\right| \ll 1$$

$$iv) \left|\frac{\overline{D}^2\delta\sigma}{\overline{\Theta}}\right| \ll 1 \quad , \ v) \left|\frac{\overline{D}^2\phi}{\kappa\left(\bar{\rho}+\bar{p}\right)-2\overline{K}}\right| \ll 1$$

$$vi) \left|\frac{\overline{D}^2\psi}{\kappa\bar{\rho}+3\overline{K}}\right| \ll 1 \quad , \quad vii) \left|\frac{\overline{D}^2\psi}{\kappa\bar{p}+\overline{K}}\right| \ll 1$$

$$viii) \left|\frac{\left(\overline{D}^2+3\overline{K}\right)\psi}{\overline{K}}\right| \ll 1 \quad .$$

- but... what if...... $\overline{K}=0$ or $\overline{\Theta}=0$

[7] Vitenti S.D.P. and Pinto-Neto, N. - *Phys. Rev.* **D 85**, 023524 (2012)

## Cosmological Perturbations

### Background dynamics

- variables: $\overline{\Theta}$, $\bar{\rho}$, $\bar{p}$, $\overline{K}$

- Equations

$$\dot{\overline{\Theta}} + \frac{3\kappa}{2}\left(\bar{\rho} + \bar{p}\right) - 3\overline{K} = 0$$

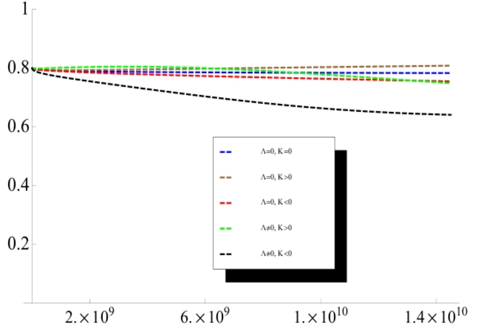
$$\frac{\overline{\Theta}^2}{3} - \kappa\bar{\rho} + 3\overline{K} = 0$$

$$\dot{\bar{\rho}} + \overline{\Theta}\left(\bar{\rho} + \bar{p}\right) = 0$$

- NEC condition is always valid, i.e. $\bar{\rho} + \bar{p} > 0$

- $\overline{m}(t) \equiv \frac{1}{2}\sqrt{|\dot{\overline{\Theta}}| + \overline{\Theta}^2} \approx max\left(\overline{\Theta}, \bar{\rho}, \bar{p}, \overline{K}\right)$

- $\overline{m}(t)$ is never zero




($\Lambda$=0, $\overline{K}$=0), ($\Lambda$=0, $\overline{K}$>0),($\Lambda$=0, $\overline{K}$<0),($\Lambda \neq$0, $\overline{K}$>0), ($\Lambda \neq$0, $\overline{K}$<0)

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# General Conditions

- metric expansion $|\delta g_{\mu\nu}| \ll 1$

$$|\phi| \ll 1\ ,\ |\psi| \ll 1\ ,\ |\overline{D}_\mu B| \ll 1\ ,\ |\overline{D}_{ij}\mathcal{E}| \ll 1$$

associated velocities expansion

$$i)\ \left|\frac{\delta\Theta}{\overline{m}}\right| \ll 1\ ,\ ii)\ \left|\frac{\overline{D}^2\psi}{\overline{m}}\right| \ll 1\ ,\ iii)\ \left|\frac{\overline{D}^2\delta\sigma}{\overline{m}}\right| \ll 1\ ,\ iv)\ \left|\frac{\overline{D}^2\phi}{\overline{m}}\right| \ll 1\ ,$$

- energy-momentum expansion...

$$|\overline{D}_\mu\mathcal{V}| \ll 1\ ,\ \ |\overline{D}^2\Pi^{\mathsf{s}}| \ll 1\ ,\ \left|\frac{\delta\rho}{\overline{m}}\right| \ll 1,\ \left|\frac{\delta p}{\overline{m}}\right| \ll 1$$

## General Conditions

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Breaking down by structure formation...

- Can FLRW be valid even though $\frac{\delta\rho}{\bar{\rho}} \gg 1$?
- Perturbed Einstein Equations $\delta\mathcal{R} = -\frac{4}{a^2}\left(\overline{D}^2 + 3\overline{K}\right)\psi$

$$Eq_{(00)} : \quad \frac{\overline{\Theta}\delta\Theta}{3} + \frac{\delta\mathcal{R}}{4} = \frac{\kappa}{2}\delta\rho$$

$$Eq_{(0i)} : \quad \delta\Theta - \left(\overline{D}^2 + 3\overline{K}\right)\delta\sigma = \frac{3}{2}\kappa\left(\bar{\rho} + \bar{p}\right)\mathcal{V}$$

- Newtonian Cosmology $(\overline{\Theta} = 0\,,\ \overline{K} = 0)$

$$\overline{D}^2\psi = \frac{\kappa}{2}\delta\rho$$

  - If

$$\delta\rho \gg 1 \qquad \Rightarrow \qquad \overline{D}^2\psi \gg 1$$

  - but

$$\overline{D}^2\psi \approx \lambda^{-2}\psi$$

  - it seems possible for small scales

## General Conditions

Breaking down by structure formation...

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$$Eq_{(00)}: \quad \frac{\bar{\Theta}\delta\Theta}{3} + \frac{\delta\mathcal{R}}{4} = \frac{\kappa}{2}\delta\rho$$

$$Eq_{(0i)}: \quad \delta\Theta - \left(\bar{D}^2 + 3\bar{K}\right)\delta\sigma = \frac{3}{2}\kappa\left(\bar{\rho} + \bar{p}\right)\mathcal{V}$$

- Expanding Universe $(\bar{\Theta} \neq 0)$ $\delta\rho^{(\mathrm{gi})} = \delta\rho + \dot{\bar{\rho}}\mathcal{V}$

$$Eq_{(00)} - \frac{\bar{\Theta}}{3}Eq_{(0i)} \quad \Rightarrow \quad \bar{D}_K^2\Psi = \frac{\kappa}{2}\delta\rho^{(\mathrm{gi})}$$

  - If

$$\frac{\delta\rho^{(\mathrm{gi})}}{\bar{\rho}} \gg 1 \quad \Rightarrow \quad \frac{\bar{D}^2\Psi}{\bar{\rho}} \gg 1$$

  - but

$$ii)\ \left|\frac{\bar{D}^2\psi}{\overline{m}}\right| \ll 1$$

  - expansion introduce a length scale

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# Back-reaction

There are three main approaches

1. perturbative approach from a fiducial background metric - Green and Wald[8]

$$\delta g_{\mu\nu} = h^{(L)}_{\mu\nu} + h^{(S)}_{\mu\nu}$$

where $h^{(S)}_{\mu\nu}$ is the "short wave-length" contribution

2. average process preserving the general covariance - R. M. Zalaletdinov [9]

- introduction of bilocal operators $\mathcal{A}^{\mu}{}_{\nu}(x,x')$ to form bitensor

$$\mathcal{T}^{\mu}{}_{\nu}(x,x') \equiv \mathcal{A}^{\mu}{}_{\alpha}(x,x')\, T^{\alpha}{}_{\beta}(x')\mathcal{A}^{\beta}{}_{\nu}(x,x')$$

the mean is defined as

$$\bar{T}^{\mu}{}_{\nu}(x) \equiv \frac{1}{V}\int_{\Sigma} \mathrm{d}^4x' \,\sqrt{-g(x')}\mathcal{T}^{\mu}{}_{\nu}(x,x') \qquad V \equiv \int_{\Sigma} \mathrm{d}^4x' \,\sqrt{-g(x')}$$

[8] Green, S. R. and Wald, R. M. - *Phys. Rev.* **D83**, 084020 (2011) and *Phys. Rev.* **D85**, 063512 (2012).
[9] Zalaletdinov, R. M. - *Gen. Rel. Grav.* **24**, 1015 (1992) and *Bull. Astron. Soc. India* **25**, 401 (1997)

# Back-reaction

3. spatial averages based on the $3+1$ decomposition - T. Buchert [10]

$$\rho\ ,\ \theta\ ,\ \sigma \equiv \sqrt{\tfrac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}}$$

- For a dust FLRW the dynamics becomes

$$\left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 = \frac{8\pi G_{\rm N}}{3}\langle\rho\rangle - \frac{1}{6}\langle\mathcal{R}\rangle - \frac{1}{6}\mathcal{Q}$$

$$\frac{\ddot{\bar{a}}}{\bar{a}} = -\frac{4\pi G_{\rm N}}{3}\langle\rho\rangle + \frac{1}{3}\mathcal{Q}$$

where

$$\bar{a} \equiv \sqrt[3]{\frac{\mathcal{V}(t)}{\mathcal{V}_0}} \qquad \mathcal{V}(t) \equiv \int_{\mathcal{D}} {\rm d}^3x\sqrt{{}^3g}$$

$$\mathcal{Q} = \frac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - 2\langle\sigma^2\rangle \qquad \langle\mathcal{R}\rangle = \frac{1}{\mathcal{V}(t)}\int_{\mathcal{D}} {\rm d}^3x\sqrt{{}^3g}\,\mathcal{R}(x,t)$$

- $\mathcal{Q}$ plays the role of an effective dark energy (might contribute up to 30% [11])

[10] Buchert T., *Class. Quantum Grav.* **28**, 164007 (2011), Buchert T., *Gen. Rel. Grav.* **32**, 105 (2000)

[11] Wilshire D. L., - *Phys. Rev. Lett.* **99**, 251101 (2007)
Li N. and Schwarz D. J., - *Phys. Rev.* **D76**, 083011 (2007) and *Phys. Rev.* **D78**, 083531 (2008).

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## Concluding...

- Primordial Universe
  - entropy perturbations, vector modes, distinguished signature of a bounce, etc...
- Late Universe
  - how to account for back-reaction?
  - David Wiltshire.... Cosmological Equivalence Principles?
  - higher order perturbations...
  - observations...
  - light geodesics in vacuum $\times$ expanding FLRW universe

$$Weyl \ \times \ Ricci$$