Charge – size inequality in General Relativity

Marcelo E. Rubio

Facultad de Matemática, Astronomía y Física
Universidad Nacional de Córdoba – (5000) Córdoba, Argentina

II Argentinian–Brazilian Meeting
on Gravitation, Astrophysics, and Cosmology

Buenos Aires, Argentina – April, 2014

- **K–N family of solutions** $\rightarrow$ 3 parameters: M, q, J.
- Many physical sistems can be characterized by a finite quantity of parameters, that can be related by

Geometric inequalities: relate quantities that have both a relevant physical motivation and a geometrical definition.

- These inequalitites tell us information about the evolution of the sistem (grav. collapse $\rightarrow$ rotating black hole).
- *Example:* Positivity of mass. The total mass m of an isolated system is $m \geq 0$ ($= 0$ only if the ST is *flat*).

- **K–N family of solutions** $\rightarrow$ 3 parameters: M, q, J.
- Many physical sistems can be characterized by a finite quantity of parameters, that can be related by

Geometric inequalities: relate quantities that have both a relevant physical motivation and a geometrical definition.

- These inequalitites tell us information about the evolution of the sistem (grav. collapse $\rightarrow$ rotating black hole).
- *Example:* Positivity of mass. The total mass m of an isolated system is $m \geq 0$ ($= 0$ only if the ST is *flat*).

- **K–N family of solutions** $\rightarrow$ 3 parameters: M, q, J.
- Many physical sistems can be characterized by a finite quantity of parameters, that can be related by

Geometric inequalities: relate quantities that have both a relevant physical motivation and a geometrical definition.

- These inequalitites tell us information about the evolution of the sistem (grav. collapse $\rightarrow$ rotating black hole).
- *Example:* Positivity of mass. The total mass m of an isolated system is $m \geq 0$ ($= 0$ only if the ST is *flat*).

- **K–N family of solutions** $\rightarrow$ 3 parameters: M, q, J.
- Many physical sistems can be characterized by a finite quantity of parameters, that can be related by

Geometric inequalities: relate quantities that have both a relevant physical motivation and a geometrical definition.

- These inequalitites tell us information about the evolution of the sistem (grav. collapse $\rightarrow$ rotating black hole).
- *Example:* POSITIVITY OF MASS. The total mass m of an isolated system is $m \geq 0$ ($= 0$ only if the ST is *flat*).

Recent results

- [Dain, 2007] **Mass–angular momentum inequality for axially symmetric BH**

$$m \geq \sqrt{J}.$$

- [Dain & Reiris, 2011] **Area–angular momentum inequality for axially symmetric BH**

$$A \geq 8\pi|J|.$$

- [Dain & Jaramillo, 2012] **Area–charge inequality for dynamic black holes**

$$A \geq 4\pi Q^2.$$

Recent results

- [Dain, 2007] **Mass–angular momentum inequality for axially symmetric BH**

$$m \geq \sqrt{J}.$$

- [Dain & Reiris, 2011] **Area–angular momentum inequality for axially symmetric BH**

$$A \geq 8\pi |J|.$$

- [Dain & Jaramillo, 2012] **Area–charge inequality for dynamic black holes**

$$A \geq 4\pi Q^2.$$

Recent results

- [Dain, 2007] **Mass–angular momentum inequality for axially symmetric BH**

$$m \geq \sqrt{J}.$$

- [Dain & Reiris, 2011] **Area–angular momentum inequality for axially symmetric BH**

$$A \geq 8\pi|J|.$$

- [Dain & Jaramillo, 2012] **Area–charge inequality for dynamic black holes**

$$A \geq 4\pi Q^2.$$

The Conjecture

- PROBLEM: ¿Is there any universal relation between the **electric charge** that can store a body and some measure of it **size**?
- **Conjecture:** CHARGE $\leq$ SIZE $\Rightarrow$ *A charged object has a minimum size given by it interior charge.*
- MOTIVATIONS:
  - Generalization of the area–charge inequality for BH.
  - [S. THORNE, 1972] **Hoop conjecture:** *"Black holes with horizons form when and only when a mass $\mathcal{M}$ gets compacted into a region whose circumference in every direction is* $\mathcal{C} \leq 4\pi\mathcal{M}$*".*

The Conjecture

- PROBLEM: ¿Is there any universal relation between the **electric charge** that can store a body and some measure of it **size**?
- **Conjecture:** CHARGE $\leq$ SIZE $\Rightarrow$ *A charged object has a minimum size given by it interior charge.*
- MOTIVATIONS:
  - Generalization of the area–charge inequality for BH.
  - [S. THORNE, 1972] **Hoop conjecture:** *"Black holes with horizons form when and only when a mass $\mathcal{M}$ gets compacted into a region whose circumference in every direction is $\mathcal{C} \leq 4\pi\mathcal{M}$".*

The Conjecture

- PROBLEM: ¿Is there any universal relation between the **electric charge** that can store a body and some measure of it **size**?
- **Conjecture:** CHARGE $\leq$ SIZE $\Rightarrow$ *A charged object has a minimum size given by it interior charge.*
- MOTIVATIONS:
  - Generalization of the area–charge inequality for BH.
  - [S. THORNE, 1972] **Hoop conjecture:** *"Black holes with horizons form when and only when a mass $\mathcal{M}$ gets compacted into a region whose circumference in every direction is $\mathcal{C} \leq 4\pi\mathcal{M}$".*

The Conjecture

- PROBLEM: ¿Is there any universal relation between the **electric charge** that can store a body and some measure of it **size**?
- **Conjecture:** CHARGE $\leq$ SIZE $\Rightarrow$ *A charged object has a minimum size given by it interior charge.*
- MOTIVATIONS:
  - Generalization of the area–charge inequality for BH.
  - [S. THORNE, 1972] **Hoop conjecture:** *"Black holes with horizons form when and only when a mass $\mathcal{M}$ gets compacted into a region whose circumference in every direction is $\mathcal{C} \leq 4\pi\mathcal{M}$".*

- If there are not symmetries... $\Rightarrow$ How to define *size*?
- In spherical symmetry, there are at least two measures:
  - **Area radius:** r_A such that, given an arbitrary geometry and coordinates $\{x^i\}$,

$$A(S) = 4\pi r_A^2, \quad A(S) := \int_S \sqrt{\det[g_{ab}]} dx_1 dx_2.$$

  - **Geodesic length:** proper distance between the boundary and the center of the spherical body.
- In flat geometry, $r_A = \ell$.

- If there are not symmetries... $\Rightarrow$ How to define *size*?
- In spherical symmetry, there are at least two measures:
  - **Area radius:** r_A such that, given an arbitrary geometry and coordinates $\{x^i\}$,

$$A(S) = 4\pi r_A^2, \quad A(S) := \int_S \sqrt{\det[g_{ab}]}dx_1 dx_2.$$

  - **Geodesic length:** proper distance between the boundary and the center of the spherical body.
- In flat geometry, $r_A = \ell$.

Spherical Symmetry

- If there are not symmetries... $\Rightarrow$ How to define *size*?
- In spherical symmetry, there are at least two measures:
  - **Area radius:** r_A such that, given an arbitrary geometry and coordinates $\{x^i\}$,

$$A(S) = 4\pi r_A^2, \quad A(S) := \int_S \sqrt{\det[g_{ab}]} dx_1 dx_2.$$

  - **Geodesic length:** proper distance between the boundary and the center of the spherical body.
- In flat geometry, $r_A = \ell$.

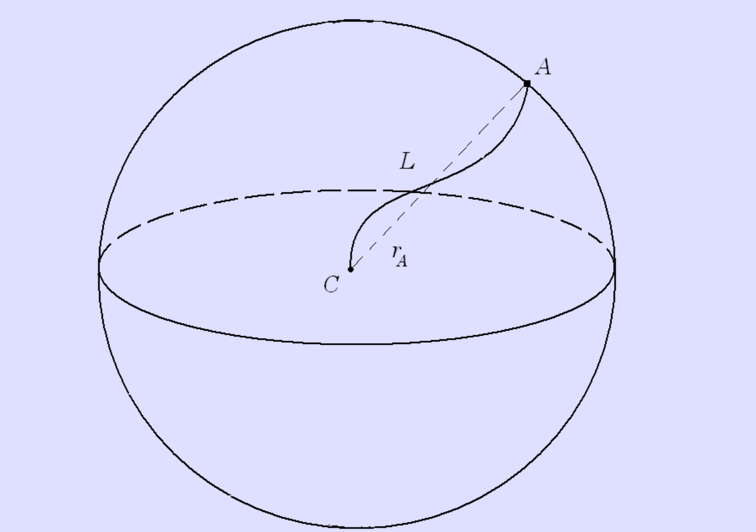
Spherical Symmetry

- If there are not symmetries... $\Rightarrow$ How to define *size*?
- In spherical symmetry, there are at least two measures:
  - **Area radius:** r_A such that, given an arbitrary geometry and coordinates $\{x^i\}$,

$$A(S) = 4\pi r_A^2, \quad A(S) := \int_S \sqrt{\det[g_{ab}]} dx_1 dx_2.$$

  - **Geodesic length:** proper distance between the boundary and the center of the spherical body.
- In flat geometry, $r_A = \ell$.

Proper distance vs. Area radius



The Conjecture in spherical symmetry

We study evidences or a counterexample of the following

Conjecture

Let Ω be an arbitrary sphere contained in an asymptotically flat initial data of field equations, such that satisfies the DEC and whose proper length is ℓ. Supose that inside Ω there is electric charge, $\mathcal{Q}$. Then the inequality

$$\mathcal{Q} \leq \ell$$

holds.

Equations

- **Initial data** $\Rightarrow \Sigma = \mathbb{R}^3$, $\mathcal{K}_{ij}|_\Sigma = 0$ (*time symmetric i.d*)
- **Source:** $\Omega \subset \Sigma \Rightarrow \mu_M,\ \rho;$
- **Constraints:**

$$\mathcal{R} = 16\pi\mu_M + 2\mathcal{E}^i\mathcal{E}_i, \quad \mathcal{D}_i\mathcal{E}^i = 4\pi\rho.$$

- **DEC** $\Rightarrow \mathcal{R} \geq 0$
- To begin, we can assume that $\mu_M = 0$.
  Next step: $\mu_M > 0$.

Equations

- **Initial data** $\Rightarrow \Sigma = \mathbb{R}^3$, $\mathcal{K}_{ij}|_{\Sigma} = 0$ (*time symmetric i.d*)
- **Source:** $\Omega \subset \Sigma \Rightarrow \mu_M$, ρ;
- **Constraints:**

$$\mathcal{R} = 16\pi\mu_M + 2\mathcal{E}^i\mathcal{E}_i, \quad \mathcal{D}_i\mathcal{E}^i = 4\pi\rho.$$

- **DEC** $\Rightarrow \mathcal{R} \geq 0$
- To begin, we can assume that $\mu_M = 0$.
  Next step: $\mu_M > 0$.

Equations

- **Initial data** $\Rightarrow \Sigma = \mathbb{R}^3$, $\mathcal{K}_{ij}|_\Sigma = 0$ (*time symmetric i.d*)
- **Source:** $\Omega \subset \Sigma \Rightarrow \mu_M$, ρ;
- **Constraints:**

$$\mathcal{R} = 16\pi\mu_M + 2\mathcal{E}^i\mathcal{E}_i, \quad \mathcal{D}_i\mathcal{E}^i = 4\pi\rho.$$

- **DEC** $\Rightarrow \mathcal{R} \geq 0$
- To begin, we can assume that $\mu_M = 0$.
  Next step: $\mu_M > 0$.

Equations

- **Initial data** $\Rightarrow \Sigma = \mathbb{R}^3$, $\mathcal{K}_{ij}|_\Sigma = 0$ (*time symmetric i.d*)
- **Source:** $\Omega \subset \Sigma \Rightarrow \mu_M,\ \rho;$
- **Constraints:**

$$\mathcal{R} = 16\pi\mu_M + 2\mathcal{E}^i\mathcal{E}_i, \quad \mathcal{D}_i\mathcal{E}^i = 4\pi\rho.$$

- **DEC** $\Rightarrow \mathcal{R} \geq 0$
- To begin, we can assume that $\mu_M = 0$.
  Next step: $\mu_M > 0$.

Equations

- **Initial data** $\Rightarrow \Sigma = \mathbb{R}^3$, $\mathcal{K}_{ij}|_\Sigma = 0$ (*time symmetric i.d*)
- **Source:** $\Omega \subset \Sigma \Rightarrow \mu_M$, ρ;
- **Constraints:**

$$\mathcal{R} = 16\pi\mu_M + 2\mathcal{E}^i\mathcal{E}_i, \quad \mathcal{D}_i\mathcal{E}^i = 4\pi\rho.$$

- **DEC** $\Rightarrow \mathcal{R} \geq 0$
- To begin, we can assume that $\mu_M = 0$.
  Next step: $\mu_M > 0$.

Conformal metric

- The metric in conformal coordinates results

$$\boxed{ds^2 = \Phi^4(r)\left[dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right)\right]},$$

  where $\Phi(r) > 0$. **Plane radius:** r

- The size measures are given by

$$\ell(r) = \int_0^r \Phi^2(t)dt, \quad r_A(r) = r\Phi^2(r)$$

- $\mathcal{R} = -\frac{8}{\Phi^5}\Delta\Phi \Rightarrow \Delta\Phi \leq 0 \Rightarrow \boxed{\ell \geq r_A}$

Conformal metric

- The metric in conformal coordinates results

$$\boxed{ds^2 = \Phi^4(r)\left[dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right)\right]},$$

where $\Phi(r) > 0$. **Plane radius:** r

- The size measures are given by

$$\ell(r) = \int_0^r \Phi^2(t)dt, \quad r_A(r) = r\Phi^2(r)$$

- $\mathcal{R} = -\frac{8}{\Phi^5}\Delta\Phi \Rightarrow \Delta\Phi \leq 0 \Rightarrow \boxed{\ell \geq r_A}$

Conformal metric

- The metric in conformal coordinates results

$$\boxed{ds^2 = \Phi^4(r)\left[dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right)\right]},$$

where $\Phi(r) > 0$. **Plane radius:** r

- The size measures are given by

$$\ell(r) = \int_0^r \Phi^2(t)dt, \quad r_A(r) = r\Phi^2(r)$$

- $\mathcal{R} = -\frac{8}{\Phi^5}\Delta\Phi \Rightarrow \Delta\Phi \leq 0 \Rightarrow \boxed{\ell \geq r_A}$

Conformal metric

- Let's define

$$\tilde{\mathcal{E}}^i(r) := \Phi^6 \mathcal{E}^i(r), \quad \tilde{\rho}(r) := \Phi^6 \rho(r),$$

such that

$$\tilde{\mathcal{D}}^i \tilde{\mathcal{E}}_i = 4\pi\tilde{\rho}; \quad \mathcal{E}^2 = \frac{\tilde{\mathcal{E}}^2}{\Phi^8}.$$

- The constraint equations become

$$\boxed{\tilde{\mathcal{D}}_i \tilde{\mathcal{E}}^i = 4\pi\tilde{\rho}; \quad \Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}}.$$

- Q is *conformally invariant* under $\rho \to \tilde{\rho}$.

Conformal metric

- Let's define

$$\tilde{\mathcal{E}}^i(r) := \Phi^6 \mathcal{E}^i(r), \quad \tilde{\rho}(r) := \Phi^6 \rho(r),$$

such that

$$\tilde{\mathcal{D}}^i \tilde{\mathcal{E}}_i = 4\pi\tilde{\rho}; \quad \mathcal{E}^2 = \frac{\tilde{\mathcal{E}}^2}{\Phi^8}.$$

- The constraint equations become

$$\boxed{\tilde{\mathcal{D}}_i \tilde{\mathcal{E}}^i = 4\pi\tilde{\rho}; \quad \Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}}.$$

- Q is *conformally invariant* under $\rho \to \tilde{\rho}$.

Conformal metric

- Let's define

$$\tilde{\mathcal{E}}^i(r) := \Phi^6 \mathcal{E}^i(r), \quad \tilde{\rho}(r) := \Phi^6 \rho(r),$$

such that

$$\tilde{\mathcal{D}}^i \tilde{\mathcal{E}}_i = 4\pi\tilde{\rho}; \quad \mathcal{E}^2 = \frac{\tilde{\mathcal{E}}^2}{\Phi^8}.$$

- The constraint equations become

$$\boxed{\tilde{\mathcal{D}}_i \tilde{\mathcal{E}}^i = 4\pi\tilde{\rho}; \quad \Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}}.$$

- Q is *conformally invariant* under $\rho \to \tilde{\rho}$.

Conformal metric

- Let's define

$$\tilde{\mathcal{E}}^i(r) := \Phi^6 \mathcal{E}^i(r), \quad \tilde{\rho}(r) := \Phi^6 \rho(r),$$

such that

$$\tilde{\mathcal{D}}^i \tilde{\mathcal{E}}_i = 4\pi\tilde{\rho}; \quad \mathcal{E}^2 = \frac{\tilde{\mathcal{E}}^2}{\Phi^8}.$$

- The constraint equations become

$$\boxed{\tilde{\mathcal{D}}_i \tilde{\mathcal{E}}^i = 4\pi\tilde{\rho}; \quad \Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}}.$$

- $\mathcal{Q}$ is <u>*conformally invariant*</u> under $\rho \to \tilde{\rho}$.

Sub/super-solutions

Definition

Consider the problem

$$\Delta u = f(\mathbf{x}, u), \quad u|_{\partial\Omega} = g, \tag{1}$$

where Ω is bounded.

- u_+ is a **super-solution** of u if

$$\Delta u_+ \leq f(\mathbf{x}, u_+), \quad u_+|_{\partial\Omega} = g.$$

- Similarly, u_- is a **sub-solution** of u if

$$\Delta u_- \geq f(\mathbf{x}, u_-), \quad u_-|_{\partial\Omega} = g.$$

Sub/super-solutions

Definition

Consider the problem

$$\Delta u = f(\mathbf{x}, u), \quad u|_{\partial\Omega} = g, \tag{1}$$

where Ω is bounded.

- u_+ is a **super-solution** of u if

$$\Delta u_+ \leq f(\mathbf{x}, u_+), \quad u_+|_{\partial\Omega} = g.$$

- Similarly, u_- is a **sub-solution** of u if

$$\Delta u_- \geq f(\mathbf{x}, u_-), \quad u_-|_{\partial\Omega} = g.$$

Sub/super-solutions

Definition

Consider the problem

$$\Delta u = f(\mathbf{x}, u), \quad u|_{\partial\Omega} = g, \tag{1}$$

where Ω is bounded.

- u_+ is a **super-solution** of u if

$$\Delta u_+ \leq f(\mathbf{x}, u_+), \quad u_+|_{\partial\Omega} = g.$$

- Similarly, u_- is a **sub-solution** of u if

$$\Delta u_- \geq f(\mathbf{x}, u_-), \quad u_-|_{\partial\Omega} = g.$$

- The following theorem gives meaning to the previous definition.

Theorem

Let u_+ *y* u_- *be a super-solution and a sub-solution to the problem*

$$\Delta u = f(\mathbf{x}, u), \quad u|_{\partial\Omega} = g.$$

Then, there exists a solution u *to the problem, and*

$$u_- \leq u \leq u_+.$$

- If $\Delta u = f$ and $f \leq 0 \Rightarrow u \geq 0$.
- If $f \leq 0$, the solution u is unique.

- The following theorem gives meaning to the previous definition.

Theorem

Let u_+ y u_- be a super-solution and a sub-solution to the problem

$$\Delta u = f(\mathbf{x}, u), \quad u|_{\partial\Omega} = g.$$

Then, there exists a solution u to the problem, and

$$u_- \leq u \leq u_+.$$

- If $\Delta u = f$ and $f \leq 0 \Rightarrow u \geq 0$.
- If $f \leq 0$, the solution u is unique.

Sub/super-solutions

- The following theorem gives meaning to the previous definition.

Theorem

Let u_+ y u_- be a super-solution and a sub-solution to the problem

$$\Delta u = f(\mathbf{x}, u), \quad u|_{\partial\Omega} = g.$$

Then, there exists a solution u to the problem, and

$$u_- \leq u \leq u_+.$$

- If $\Delta u = f$ and $f \leq 0 \Rightarrow u \geq 0$.
- If $f \leq 0$, the solution u is unique.

Sub/super-solutions

- The following theorem gives meaning to the previous definition.

Theorem

Let u_+ *y* u_- *be a super-solution and a sub-solution to the problem*

$$\Delta u = f(\mathbf{x}, u), \quad u|_{\partial\Omega} = g.$$

Then, there exists a solution u *to the problem, and*

$$u_- \leq u \leq u_+.$$

- If $\Delta u = f$ and $f \leq 0 \Rightarrow u \geq 0$.
- If $f \leq 0$, the solution u is <u>unique</u>.

Results

Problem

Solve

$$\Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}, \quad \lim_{r\to\infty} \Phi = 1;$$

for a sphere of plane radius R with constant conformal density

$$\tilde{\rho}(r) = \begin{cases} \tilde{\rho}_0, & r \leq R \\ 0, & r > R \end{cases}$$

- u such that $\Phi = 1 + u \Rightarrow \Delta u = -\tilde{\mathcal{E}}^2(4(1+u)^3)^{-1}$.
- The function u_+ that satisfies the problem

$$\Delta u_+ = -\frac{\tilde{\mathcal{E}}^2}{4}, \quad \lim_{r\to\infty} u_+ = 0,$$

is a **super-solution** of $u \Rightarrow u \leq u_+$.

Super-solution for area radius

Problem

Solve

$$\Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}, \quad \lim_{r\to\infty} \Phi = 1;$$

for a sphere of plane radius R with constant conformal density

$$\tilde{\rho}(r) = \begin{cases} \tilde{\rho}_0, & r \leq R \\ 0, & r > R \end{cases}$$

- u such that $\Phi = 1 + u \Rightarrow \Delta u = -\tilde{\mathcal{E}}^2(4(1+u)^3)^{-1}$.
- The function u_+ that satisfies the problem

$$\Delta u_+ = -\frac{\tilde{\mathcal{E}}^2}{4}, \quad \lim_{r\to\infty} u_+ = 0,$$

is a **super-solution** of $u \Rightarrow u \leq u_+$.

Super-solution for area radius

Problem

Solve

$$\Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}, \quad \lim_{r\to\infty} \Phi = 1;$$

for a sphere of plane radius R with constant conformal density

$$\tilde{\rho}(r) = \begin{cases} \tilde{\rho}_0, & r \leq R \\ 0, & r > R \end{cases}$$

- u such that $\Phi = 1 + u \Rightarrow \Delta u = -\tilde{\mathcal{E}}^2(4(1+u)^3)^{-1}$.
- The function u_+ that satisfies the problem

$$\Delta u_+ = -\frac{\tilde{\mathcal{E}}^2}{4}, \quad \lim_{r\to\infty} u_+ = 0,$$

is a **super-solution** of $u \Rightarrow u \leq u_+$.

Super-solution for area radius

Problem

Solve

$$\Delta\Phi = -\frac{\tilde{\mathcal{E}}^2}{4\Phi^3}, \quad \lim_{r\to\infty}\Phi = 1;$$

for a sphere of plane radius R with constant conformal density

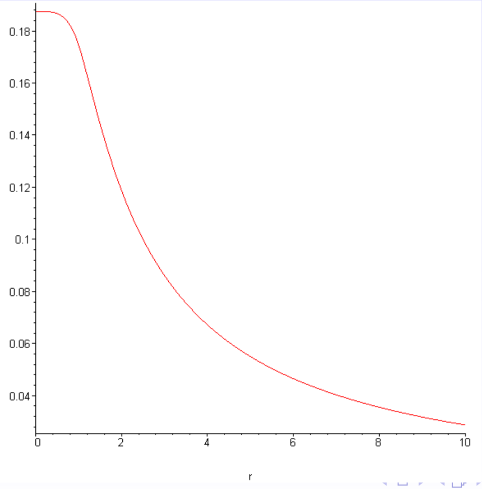
$$\tilde{\rho}(r) = \begin{cases} \tilde{\rho}_0, & r \leq R \\ 0, & r > R \end{cases}$$

- u such that $\Phi = 1 + u \Rightarrow \Delta u = -\tilde{\mathcal{E}}^2(4(1+u)^3)^{-1}$.
- The function u_+ that satisfies the problem

$$\Delta u_+ = -\frac{\tilde{\mathcal{E}}^2}{4}, \quad \lim_{r\to\infty} u_+ = 0,$$

is a **super-solution** of $u \Rightarrow u \leq u_+$.

The function u_+, $\tilde{\rho}_0 = \frac{3}{4\pi}$



Super-solution for area radius

- An upper bound for r_A in terms of u_+ is

$$r_A^+ := r\Phi_+^2 \geq r_A = r\Phi^2, \quad \Phi_+ = 1 + u_+.$$

Theorem

Let r_A^+ be the above upper-bound for the area radius and $\mathcal{Q}(r)$ the total charge inside a sphere of plane radius r and constant conformal density. Then the inequality

$$r_A^+ \geq \mathcal{Q},$$

holds $\forall r \geq 0$.

- Φ_+ is monotonically decreasing, then

$$\ell_+(r) := \int_0^r \Phi_+^2(t)dt \quad \Rightarrow \quad \boxed{\ell_+ \geq r_A^+},$$

so the above theorem is also true for ℓ_+.

Super-solution for area radius

- An upper bound for r_A in terms of u_+ is

$$r_A^+ := r\Phi_+^2 \geq r_A = r\Phi^2, \quad \Phi_+ = 1 + u_+.$$

Theorem

Let r_A^+ be the above upper-bound for the area radius and $\mathcal{Q}(r)$ the total charge inside a sphere of plane radius r and constant conformal density. Then the inequality

$$r_A^+ \geq \mathcal{Q},$$

holds $\forall r \geq 0$.

- Φ_+ is monotonically decreasing, then

$$\ell_+(r) := \int_0^r \Phi_+^2(t)dt \quad \Rightarrow \quad \boxed{\ell_+ \geq r_A^+},$$

so the above theorem is also true for ℓ_+.

Super-solution for area radius

- An upper bound for r_A in terms of u_+ is

$$r_A^+ := r\Phi_+^2 \geq r_A = r\Phi^2, \quad \Phi_+ = 1 + u_+.$$

Theorem

Let r_A^+ be the above upper-bound for the area radius and $\mathcal{Q}(r)$ the total charge inside a sphere of plane radius r and constant conformal density. Then the inequality

$$r_A^+ \geq \mathcal{Q},$$

holds $\forall r \geq 0$.

- Φ_+ is monotonically decreasing, then

$$\ell_+(r) := \int_0^r \Phi_+^2(t)dt \quad \Rightarrow \quad \boxed{\ell_+ \geq r_A^+},$$

so the above theorem is also true for ℓ_+.

Sub-solution for area radius

- We then prove the following

Lemma

Let be u_- the solution of the problem

$$\Delta u_- = -\frac{\tilde{\mathcal{E}}^2}{(1+\overline{u}_+)^3}, \quad \lim_{r\to\infty} u_- = 0,$$

where $\overline{u}_+$ is the maximum value of u_+. Then u_- is a **sub-solution** of u, and $u_- \leq u$.

- A lower bound for r_A is

$$r_A^- := r\Phi_-(r)^2 \leq r\Phi(r)^2 = r_A, \quad \Phi_-(r) := 1 + u_-.$$

Sub-solution for area radius

- We define the variables

$$x(r) := \alpha r^2, \quad \Lambda := \alpha^2 R^4 = x^2(R), \quad \alpha := \frac{4}{3}\pi\tilde{\rho}_0$$

- For $r \leq R$, we have $0 < x \leq \sqrt{\Lambda}$ and

$$\begin{aligned}\frac{r_A^- - Q}{r} &= \frac{1}{256\left(1+\frac{3\Lambda}{16}\right)^6}\left(\frac{x^2}{5} - 3\Lambda\right)^2 + \frac{15\Lambda - x^2}{40\left(1+\frac{3\Lambda}{16}\right)^3} - x + 1 \\ &\geq 1 - \sqrt{\Lambda} \\ &\geq 0,\end{aligned}$$

if $\sqrt{\Lambda} \leq 1 \Leftrightarrow Q \leq R$.

- We define the variables

$$x(r) := \alpha r^2, \quad \Lambda := \alpha^2 R^4 = x^2(R), \quad \alpha := \frac{4}{3}\pi\tilde{\rho}_0$$

- For $r \leq R$, we have $0 < x \leq \sqrt{\Lambda}$ and

$$\begin{aligned}\frac{r_A^- - \mathcal{Q}}{r} &= \frac{1}{256\left(1+\frac{3\Lambda}{16}\right)^6}\left(\frac{x^2}{5} - 3\Lambda\right)^2 + \frac{15\Lambda - x^2}{40\left(1+\frac{3\Lambda}{16}\right)^3} - x + 1 \\ &\geq 1 - \sqrt{\Lambda} \\ &\geq 0,\end{aligned}$$

if $\sqrt{\Lambda} \leq 1 \Leftrightarrow \mathcal{Q} \leq R$.

- We define the variables

$$x(r) := \alpha r^2, \quad \Lambda := \alpha^2 R^4 = x^2(R), \quad \alpha := \frac{4}{3}\pi\tilde{\rho}_0$$

- For $r \leq R$, we have $0 < x \leq \sqrt{\Lambda}$ and

$$\begin{aligned}\frac{r_A^- - \mathcal{Q}}{r} &= \frac{1}{256\left(1+\frac{3\Lambda}{16}\right)^6}\left(\frac{x^2}{5} - 3\Lambda\right)^2 + \frac{15\Lambda - x^2}{40\left(1+\frac{3\Lambda}{16}\right)^3} - x + 1 \\ &\geq 1 - \sqrt{\Lambda} \\ &\geq 0,\end{aligned}$$

if $\sqrt{\Lambda} \leq 1 \Leftrightarrow \mathcal{Q} \leq R$.

Sub-solution for area radius

- The main result of this work is the following.

Theorem

Consider an asymptotically flat initial data for the Einstein-Maxwell equations containing a sphere of plane radius R with constant conformal charge density and total charge Q inside it such that

$$R \geq Q.$$

Then, the following inequality

$$r_A(r) \geq Q(r)$$

is satisfied for all $r \leq R$.

Sub-solution for area radius

- The main result of this work is the following.

Theorem

Consider an asymptotically flat initial data for the Einstein-Maxwell equations containing a sphere of plane radius R with constant conformal charge density and total charge Q inside it such that

$$R \geq Q.$$

Then, the following inequality

$$r_A(r) \geq Q(r)$$

is satisfied for all $r \leq R$.

Conclussions

- We study an universal inequality between charge and size of objects in General Relativity with spherical symmetry.
- We showed particular evidences in favor of the inequality.
- It was possible to prove the inequality inside the sphere under certain assumptions about charge distribution.

Conclussions

- We study an universal inequality between charge and size of objects in General Relativity with spherical symmetry.
- We showed particular evidences in favor of the inequality.
- It was possible to prove the inequality inside the sphere under certain assumptions about charge distribution.

Conclussions

- We study an universal inequality between charge and size of objects in General Relativity with spherical symmetry.
- We showed particular evidences in favor of the inequality.
- It was possible to prove the inequality inside the sphere under certain assumptions about charge distribution.

Future prospects

- Numerical studies of the non linear elliptic equation for the conformal factor of the initial metric.
- Prove analytically the inequality in spherical symmetry for the general case.

Future prospects

- Numerical studies of the non linear elliptic equation for the conformal factor of the initial metric.
- Prove analytically the inequality in spherical symmetry for the general case.

Thank you!